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A universal property of global spectra

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Equivariant enhancements of many important cohomology theories such as singular cohomology, topological K-theory, or various versions of bordism come to us as a 'compatible family' of G-spectra as the compact Lie group G varies. Such a compatible family of G-spectra is called a *global spectrum*, cf. [5]. In this poster, we characterize the category of global spectra using parametrized homotopy theory. This is joint work with Tobias Lenz.

The ∞-categories of (genuine) G-spaces and G-spectra for varying G come with various 'change of group' functors (restriction functors, their adjoints, the Hill–Hopkins–Ravenel norms, geometric fixed points, $\dots$), and these are important tools to construct G-spectra from simpler data or to conversely reduce questions to smaller groups in an inductive fashion. It is therefore natural to not restrict one's attention to a fixed group, but instead work *parametrized* by a suitable family of groups. Let us make this precise.

Global categories

In [4], Gepner and Henriques construct the *global orbit ∞-category* Glo whose objects are compact Lie groups and whose mapping spaces are

$$\mathrm{Map}_{\mathrm{Glo}}(H,G) \simeq \sqcup_{\alpha\colon H\to G} BC_G(\alpha), \tag{1}$$

where α runs through representatives of conjugacy classes of continuous group homomorphisms $H \to G$ and $C_G(\alpha) \subseteq G$ denotes the centralizer of the image of α.

Definition. We define the category of global categories as $\mathrm{CAT}_{\mathrm{Glo}} \coloneqq \mathrm{Fun}(\mathrm{Glo}^{\mathrm{op}}, \mathrm{CAT}_\infty)$.

The category of pointed objects in Glo is equivalent to the nerve of the topological category of compact Lie groups and continuous group homomorphisms (topologized via compact open topology). A global category $\mathcal{C}$ therefore in particular encodes a family of ∞-categories $\mathcal{C}(G)$ indexed by compact Lie groups G, together with restriction functors $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ along continuous group homomorphisms $\alpha\colon H \to G$. The higher coherences in particular whitness that inner automorphisms act via the identity on $\mathcal{C}(G)$.

There is a functor $\mathrm{Glo} \to \mathrm{Spc}$ sending a compact Lie group G to its classifying space BG, which is fully faithful when restricted to the full subcategory on compact Lie groups with abelian identity component by [7]. It is however *not* fully faithful on all of Glo.

Examples. The global category represented by the orthogonal group $O(n)$ sends a compact Lie group G to the nerve of the topological category of n-dimensional real G-representations and equivariant linear isometric isomorphisms (topologized as subspaces of $O(n)$).

To any topological category $\mathcal{C}$, one can associate a global category $N_{\mathrm{top}}(\mathcal{C}^\flat)$ of *continuous Borel objects in* $\mathcal{C}$, which sends a compact Lie group G to the nerve of the topological category of continuous G-objects in $\mathcal{C}$, and a group homomorphism $\alpha\colon H \to G$ to the restriction functor. This is functorial in the topological category $\mathcal{C}$ and moreover lifts (uniquely) to a functor $\mathrm{SymTopCat} \to \mathrm{CMon}(\mathrm{CAT}_{\mathrm{Glo}})$ from the 1-category of symmetric monoidal topological categories to symmetric monoidal global categories.

Applying the continuous Borel construction to the topological category of topological spaces and localizing levelwise at the genuine weak equivalences (those continuous G-equivariant maps $f\colon X \to Y$ such that $f^H\colon X^H \to Y^H$ is a weak equivalence for all closed subgroups $H \subseteq G$) yields a global category $\underline{\mathfrak{S}}$ which sends a compact Lie group G to the ∞-category of G-spaces.

The ∞-category of global spaces is defined as $\mathrm{Spc}_{\mathrm{gl}} \coloneqq \mathrm{Fun}(\mathrm{Glo}^{\mathrm{op}}, \mathrm{Spc})$. For a compact Lie group G, denote by $\mathrm{Orb}_G \subseteq \underline{\mathfrak{S}}(G)$ the full category on transitive G-spaces, i.e. those of the form G/H for closed subgroups $H \subseteq G$. Using 1, one can define a wide subcategory $\mathrm{Orb} \subseteq \mathrm{Glo}$ on injective group homomorphisms. For a compact Lie group G, [5] describe an equivalence $\mathrm{Orb}_G \simeq \mathrm{Orb}_{/G}$. Restriction along the forget functor $\mathrm{Orb}_G \simeq \mathrm{Orb}_{/G} \to \mathrm{Glo}_{/G} \xrightarrow{\pi_G} \mathrm{Glo}$ yields a functor $\mathrm{res}_G\colon \mathrm{Spc}_{\mathrm{gl}} \to \mathrm{PSh}(\mathrm{Orb}_G) \simeq \underline{\mathfrak{S}}(G)$.

Right Kan extension and restriction induce mutually inverse equivalences

$$\mathrm{CAT}_{\mathrm{Glo}} \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathrm{PSh}(\mathrm{Glo})^{\mathrm{op}}, \mathrm{CAT}_\infty),$$

and we will in the sequel evaluate a global ∞-category at a global space by passing to its limit extension implicitly. Parametrized homotopy theory ([2, 6]) studies the category $\mathrm{Fun}^{\mathrm{lim}}(\mathcal{B}^{\mathrm{op}}, \mathrm{CAT}_\infty)$ for a general ∞-topos $\mathcal{B}$. The framework of parametrized homotopy theory allows to formulate *genuine* versions of categorical properties such as cocompleteness and stability, which we apply to characterize equivariant and global, unstable and stable homotopy theory through universal properties.

Equivariant and global cocompleteness

[3] introduces a notion of equivariant (Orb-) cocompleteness and global (Glo-)cocompleteness for global categories. This has a natural formulation in terms of parametrized category theory, which unwinds to the following:

Definition. Let $S \in \{\mathrm{Orb}, \mathrm{Glo}\}$. A global category $\mathcal{C}$ is *S-cocomplete* if

(1) For all compact Lie groups G, $\mathcal{C}(G)$ is cocomplete and for all continuous group homomorphisms $\alpha\colon H \to G$, the restriction $\alpha^*_{\mathcal{C}}\colon \mathcal{C}(G) \to \mathcal{C}(H)$ preserves colimits.

(2) For all continuous group homomorphisms $\alpha\colon H \to G \in S$, the restriction functor $\alpha^*_{\mathcal{C}}\colon \mathcal{C}(G) \to \mathcal{C}(H)$ admits a left adjoint $\alpha^{\mathcal{C}}_!\colon \mathcal{C}(H) \to \mathcal{C}(G)$, and these satisfy the following basechange condition: For all pullbacks

$$\begin{tikzcd} P \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\beta}"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "\beta"] \\ K \arrow[r, "\alpha"] & G \end{tikzcd}$$

in $\mathrm{PSh}(\mathrm{Glo})$ with $H, K, G \in \mathrm{Glo}$ and $\alpha \in S$, restriction along $\tilde{\alpha}$ admits a left adjoint $\tilde{\alpha}^{\mathcal{C}}_!$ and the Beck-Chevalley map $\tilde{\alpha}^{\mathcal{C}}_! \circ \tilde{\beta}^*_{\mathcal{C}} \to \beta^*_{\mathcal{C}} \circ \alpha^{\mathcal{C}}_!$ is an equivalence.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ between S-cocomplete global categories is *S-cocontinuous* if it is levelwise cocontinuous (i.e. $F(G)\colon \mathcal{C}(G) \to \mathcal{D}(G)$ is cocontinuous for all compact Lie groups G), and for all $\alpha\colon H \to G \in S$, the Beck-Chevalley map $\alpha^{\mathcal{D}}_! \circ F(H) \to F(G) \circ \alpha^{\mathcal{C}}_!$ is an equivalence. We denote by $\mathrm{CAT}^{S-\mathrm{cc}}_{\mathrm{Glo}}$ the category of S-cocomplete gobal categories and S-cocontinuous functors.

By general results on parametrized categories ([6]), the forget functor $\mathrm{CAT}^{S-\mathrm{cc}}_{\mathrm{Glo}} \to \mathrm{CAT}_{\mathrm{Glo}}$ has a left adjoint.

Theorem (Lenz-B.). *The global category $\underline{\mathfrak{S}}$ of equivariant spaces constructed above is the free* Orb*-cocomplete global category on one generator.*

We identify the free globally cocomplete global category on one generator $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ with the localization of the continuous Borel construction of orthogonal spaces at the G-global equivalences from [1] and apply this to exhibit $\underline{\mathfrak{S}}$ as the full subcategory of $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ generated under Orb-colimits by $*$, which then proves the above theorem.

Remark. Using general results from parametrized category theory and Elmendorf's theorem, it is easy to see that the free Orb/Glo-cocomplete global category on $*$ evaluates to the category of G-spaces/G-global spaces at every compact Lie group G. The hard part in proving the above results is to identify the functoriality in restriction homomorphisms including all higher coherences, which is crucial if one wants to characterize classical pointset constructions through their parametrized universal properties, as we do for Thom spectra.

For $S \in \{\mathrm{Orb}, \mathrm{Glo}\}$, the category of S-cocomplete global categories and S-cocontinuous functors $\mathrm{CAT}^{S-\mathrm{cc}}_{\mathrm{Glo}}$ carries a symmetric monoidal structure (constructed analogously to the Lurie tensor product on cocomplete/presentable ∞-categories). The units are $\underline{\mathfrak{S}}$ and $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$, respectively, and the induced symmetric monoidal structures on both categories are cartesian. The left adjoint of the forget functor $\mathrm{CAT}^{S-\mathrm{cc}}_{T} \to \mathrm{CAT}_{\mathrm{Glo}}$ enhances to a symmetric monoidal functor.

Equivariant stability

Recall that the category Spc_* of pointed spaces is an idempotent in Cat^{cc}_∞ (with Lurie tensor product) and modules over Spc_* are precisely the cocomplete categories admitting a zero object. A pointed, cocomplete category $\mathcal{C}$ is stable if and only if the tensoring $S^1 \otimes -\colon \mathcal{C} \to \mathcal{C}$ is an equivalence. Analogously, every pointed, Orb-cocomplete global category $\mathcal{C}$ admits a unique $\underline{\mathfrak{S}}_*$-module structure. For every compact Lie group G, this evaluates to a left-tensoring $- \otimes -\colon \underline{\mathfrak{S}}(G)_* \times \mathcal{C}(G) \to \mathcal{C}(G)$.

Definition. A pointed, Orb-cocomplete global category $\mathcal{C}$ is *representation-stable* if for all compact Lie groups G and all finite-dimensional G-representations V, $S^V \otimes -\colon \mathcal{C}(G) \to \mathcal{C}(G)$ is an equivalence, where S^V denotes the one-point-compactification of V.

Proposition (Lenz-B.). *Let $S \in \{\mathrm{Orb}, \mathrm{Glo}\}$. The category of S-cocomplete, representation-stable global categories is a smashing localization of* $\mathrm{CAT}^{S-\mathrm{cc}}_{\mathrm{Glo}}$.

The free Orb-cocomplete, representation-stable global category can be constructed as follows: Via the cartesian symmetric monoidal structure, $\underline{\mathfrak{S}}$ enhances to a functor $\underline{\mathfrak{S}}^\times\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}^{\mathrm{cc}}_\infty)$. By pointwise tensor-inverting representation spheres in $\mathrm{CAlg}(\mathrm{CAT}^{\mathrm{cc}}_\infty)$, we obtain a symmetric monoidal global category $\underline{\mathfrak{Sp}}$ sending a compact Lie group G to the category of genuine G-spectra with the smash product symmetric monoidal structure. We show that this is the free equivariantly cocomplete, representation-stable global category.

We can also describe the free globally cocomplete, representation-stable global category: By localizing the continuous Borel construction of orthogonal spectra levelwise at the G-global equivalences from [8], one obtains a global category $\underline{\mathfrak{Sp}}_{\mathrm{gl}}$ sending a compact Lie group G to the category of G-global spectra.

Theorem (Lenz-B.). *$\underline{\mathfrak{Sp}}_{\mathrm{gl}}$ is the free globally cocomplete, representation-stable global category.*

The category of G-global spectra is *not* obtained by tensor-inverting representation spheres in the category of G-global spaces. For instance, at the trivial group, the tensor-inversion is just the ordinary stabilization of global spaces, which is different from global spectra. Only the global functor $\underline{\mathrm{Spc}}_{\mathrm{Glo},*} \to \underline{\mathfrak{Sp}}_{\mathrm{gl}} \in \mathrm{CAlg}(\mathrm{CAT}^{\mathrm{Glo}-\mathrm{cc}}_{\mathrm{Glo}})$ is characterized by inverting representation spheres. The proof of the above universal property of global spectra uses entirely different methods than the corresponding equivariant statement.

The categories $\underline{\mathfrak{Sp}}$ and $\underline{\mathfrak{Sp}}_{\mathrm{gl}}$ admit unique equivariantly/globally cocontinuous symmetric monoidal structures, and there is a unique Orb-cocontinuous, symmetric monoidal functor $i\colon \underline{\mathfrak{Sp}}^\otimes \to \underline{\mathfrak{Sp}}^\otimes_{\mathrm{gl}}$. This functor is fully faithful, and an Orb-parametrized left adjoint. For every compact Lie group G, the right adjoint of $i(G)$ is a symmetric monoidal localization. In particular, i restricts to an equivalence $\mathrm{pic}(\underline{\mathfrak{Sp}}) \simeq \mathrm{pic}(\underline{\mathfrak{Sp}}_{\mathrm{gl}})$.

(Global) equivariant Thom spectra

Using the symmetric monoidal continuous Borel construction, we can construct a commutative monoid $\underline{\mathrm{Rep}}^\oplus$ in global spaces sending a compact Lie group G to the nerve of the topological category of finite-dimensional G-representations and isomorphisms, which is a commutative monoid under direct sum, and a symmetric monoidal global functor $S^{(-)}\colon \underline{\mathrm{Rep}}^\oplus \to \underline{\mathfrak{S}}_* \subseteq \underline{\mathrm{Spc}}_{\mathrm{Glo},*}, V \mapsto S^V$. The composite $\Sigma^\infty \circ S^{(-)}\colon \underline{\mathrm{Rep}}^\oplus \to \underline{\mathfrak{Sp}}_{\mathrm{gl}}$ factors over the subgroupoid on $\otimes$-invertible objects, and hence extends uniquely to a symmetric monoidal global functor $J\colon \underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}} \to \underline{\mathfrak{Sp}}_{\mathrm{gl}}$ from the group completion of $\underline{\mathrm{Rep}}^\oplus$, which we call the *global J-homomorphism.*

For a commutative monoid $M \in \mathrm{CMon}(\mathrm{Spc}_{\mathrm{gl}})$, there exists a certain global symmetric monoidal slice category $\underline{\mathrm{Spc}}_{\mathrm{Glo}/M}$ sending G to $\mathrm{PSh}(\mathrm{Glo}_{/G})_{/\pi^*_G M}$ with the slice symmetric monoidal structure, and a symmetric monoidal global functor $y^\otimes_M\colon M \hookrightarrow \underline{\mathrm{Spc}}_{\mathrm{Glo}/M}$ exhibiting $\underline{\mathrm{Spc}}_{\mathrm{Glo}/M}$ as the symmetric monoidal global cocompletion of M, i.e. for every globally cocomplete symmetric monoidal category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{CAT}^{\mathrm{Glo}-\mathrm{cc}}_{\mathrm{Glo}})$, restriction along $y^\otimes_M$ yields an equivalence $\mathrm{Fun}^{\mathrm{Glo}-\mathrm{cc},\otimes}(\underline{\mathrm{Spc}}_{\mathrm{Glo}/M}, \mathcal{C}) \simeq \mathrm{Fun}^\otimes(M, \mathcal{C})$ between globally cocontinuous, symmetric monoidal functors $\underline{\mathrm{Spc}}_{\mathrm{Glo}/M} \to \mathcal{C}$ and symmetric monoidal functors $M \to \mathcal{C}$.

Definition. We define the global Thom space functor $\underline{\mathrm{th}}_{\mathrm{gl}}\colon \underline{\mathrm{Spc}}_{\mathrm{Glo}/\underline{\mathrm{Rep}}^\oplus} \to \underline{\mathrm{Spc}}_{\mathrm{Glo},*}$ as the unique Glo-cocontinuous symmetric monoidal extension of $S^{(-)}$, and the global Thom spectrum functor $\underline{\mathrm{Th}}_{\mathrm{gl}}\colon \underline{\mathrm{Spc}}_{\mathrm{Glo}/\underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}}} \to \underline{\mathfrak{Sp}}_{\mathrm{gl}}$ as the unique Glo-cocontinuous symmetric monoidal extension of J.

By construction, the Thom space and Thom spectrum functor fit into a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Spc}}_{\mathrm{Glo}/\underline{\mathrm{Rep}}^\oplus} \arrow[r, "\underline{\mathrm{th}}_{\mathrm{gl}}"] \arrow[d] & \underline{\mathrm{Spc}}_{\mathrm{Glo},*} \arrow[d, "\Sigma^\infty"] \\ \underline{\mathrm{Spc}}_{\mathrm{Glo}/\underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}}} \arrow[r, "\underline{\mathrm{Th}}_{\mathrm{gl}}"] & \underline{\mathfrak{Sp}}_{\mathrm{gl}} \end{tikzcd} \tag{2}$$

of symmetric monoidal global categories, where the left vertical functor is the unique symmetric monoidal, globally cocontinuous extension of $\underline{\mathrm{Rep}}^\oplus \to \underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}} \xrightarrow{y^\otimes_{\underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}}}} \underline{\mathrm{Spc}}_{\mathrm{Glo}/\underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}}}$. We learned the following interpretation of the universal property of global spectra from Gepner, Nikolaus and Schwede:

Corollary. *Diagram (2) is a pushout in* $\mathrm{CAlg}(\mathrm{CAT}^{\mathrm{Glo}-\mathrm{cc}}_{\mathrm{Glo}})$.

As $\mathrm{pic}(\underline{\mathfrak{Sp}}) \simeq \mathrm{pic}(\underline{\mathfrak{Sp}}_{\mathrm{gl}})$, the global J-homomorphism factors uniquely over a symmetric monoidal functor $J_{\mathrm{eq}}\colon \underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}} \to \underline{\mathfrak{Sp}}$. Analogously to the above, one can construct equivariant Thom space and equivariant Thom spectrum functors $\underline{\mathfrak{S}}_{/\underline{\mathrm{Rep}}^\oplus} \to \underline{\mathfrak{S}}_*$ and $\underline{\mathfrak{S}}_{/\underline{\mathrm{Rep}}^{\oplus,\mathrm{grp}}} \to \underline{\mathfrak{Sp}}$. The global Thom space and Thom spectrum functors recover the equivariant ones in two ways, and all classical equivariant Thom spectra (such as $\mathrm{mO}, \mathrm{mSO}, \mathrm{mU}, \mathrm{MO}, \mathrm{MSO}, \mathrm{MU}, \dots$) admit global refinements.

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